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The Open University
Mathematics/Science/Technology
An Inter-faculty Second Level Course
MST204 Mathematical Models and Methods

mathematical models and methods

Unit 12

Heat transfer

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1.7

Contents

Introduction

This is a modelling unit. Its subject is one which is not only of interest to many people, but also lends itself to modelling particularly well, in that even the simpler models can be made to give useful answers to practical questions. It will also give you the opportunity of using, in a practical context, some of the mathematical techniques discussed earlier in the course. If you have studied *TM281* you will find some of the topics familiar.

Study guide

This unit is intended to be read as it is written—the sections following each other in numerical order. The first four sections are normal teaching sections; Section 5 contains no new material, but consists of numerical problems somewhat more difficult and comprehensive than the exercises in the first four sections. You should attempt as many of these problems as you can without looking at the solutions. When you have done all you can (or all you have time for), read the solutions carefully—you should find them instructive.

There is no tape associated with this unit but there is a television programme. The exact stage of your study of this unit at which you watch it is not too important, except that you should have read at least Section 1 first. The ideal point to watch it is when you reach Section 3.

1 The nature of heat transfer

1.1 What is heat transfer?

This unit is about the transmission of energy from one place to another. For an example you might look at the way in which the energy produced by the sun is available to plants on earth, or the way in which some of the energy from a gas fire, intended to heat a room, escapes through the walls to heat the air outside. This kind of redistribution of energy is conventionally referred to as **heat transfer.**

Actually, this term is something of a misnomer. It leads to such misleading statements as 'heat flows from one body to another'—as though we were dealing with a material substance; a kind of fluid perhaps. A more accurate, though lengthier, title for this unit would be *Energy transfer by heating*, where 'heating' is a particular mechanism for acquiring or disposing of energy.

The essential feature of this mechanism is the existence of a temperature difference, combined with the rule that the net transfer of energy takes place *from* regions of higher temperature *to* regions of lower temperature. For example, the fact that energy is transferred from a heated room to the cooler atmosphere outside the house, through the walls, is only too clear to the householder who has to pay for the fuel which keeps the heater going. Again, the energy reaches the air in the room via, say, a central heating radiator which is itself at a temperature above that of the room.

These are just two examples of the kind of process we shall try to model in this unit, and if now and again I use the term *heat transfer*, it is because it has become generally accepted. It will not be allowed to introduce any unnecessary difficulties.

Heating is not the only way in which energy can be transferred. If you wind up a watch you are transferring energy from your muscles to the watch spring. A temperature difference is irrelevant to this operation (known as transferring energy by *working*), and it will not therefore be dealt with in this unit. However, the fact that the discussion will be restricted to energy transfers which rely on temperature differences for their driving force does not mean that our examples will lack variety. There are in fact three distinct modes of heat transfer, and I shall have something to say about each of them.

I shall develop some specific models which are widely used, sometimes to predict the rate at which energy is being transferred (or the amount of energy transferred in a given time), and sometimes to predict the way the temperature will vary inside a piece of material. First of all, however, I must say something about the three modes of transferring energy by heating.

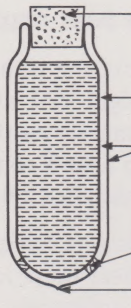
1.2 The three modes of heat transfer

The three modes of heat transfer are called **conduction**, **radiation** and **convection**. In practice they often occur in combination, but for our present purpose it is convenient to consider them one at a time.

In **conduction**, energy is transferred from one part of a piece of material to another part because the faster movement of atoms or molecules or electrons in the hotter part (i.e. the part at higher temperature) excites the atoms or molecules or electrons in the cooler part so that they move faster. In this way energy is passed on along the body from the parts at higher temperature to those at lower temperature. There is no macroscopic (i.e. large-scale) bodily movement of any part of the material. I have already mentioned one example of conduction: the loss of energy through the walls of a heated room. The inside face of the wall is at a higher temperature than the outside face and, as a result, energy passes through the wall by conduction. The same process can take place between two different pieces of material in contact; energy will pass from the hotter to the cooler.

The second mode of heat transfer is **radiation**. A familiar example of radiation is the way that energy reaches us from the sun. Unlike conduction, radiation does not require the presence of any material object. Radiation travels through empty

space. The mechanism by which the energy travels is called **electromagnetic radiation**. This course is not the place for a detailed discussion of electromagnetic radiation, but it is worth mentioning that radio and television broadcasts, light and X-rays are all examples of it. But radiation does not take place only in empty space. Every piece of material continuously dissipates part of its energy by radiation and gains energy by radiation from the other objects around it. For a given body the net rate at which it gains or loses energy by radiation depends on its temperature and that of its surroundings, as well as the nature of its surface. The higher the temperature of the body the greater the amount of energy it radiates in a given time. A highly polished surface is a poor radiator and also a poor absorber of radiation. This is why in a Thermos flask the surfaces which face each other across the vacuum space are highly polished reflectors.



The third method of energy transfer, **convection**, is associated with **fluids** (i.e. liquids and gases). Energy is transferred by virtue of the movement of the fluid. A saucepan of water on a gas ring is a suitable example. Energy from the burning gas is conducted through the bottom of the saucepan and transmitted to the parts of the water nearest the bottom of the pan, mainly by conduction. This raises the temperature of that part of the water and consequently makes it less dense. The hotter water then rises and mixes with the main body of the water, transmitting energy to it, again mainly by conduction. It is replaced near the bottom of the pan by cooler water from above and the process continues. As a result, there are **convection currents** in the water. If they are caused purely by the kind of variation in density with temperature that I described in the case of the water in the saucepan, the process is known as **free convection**. If the fluid movement, and therefore the rate of heat transfer, were to be increased by the use of a stirrer or a pump, then it would be **forced convection**.

1.3 SI units concerned with heat transfer

Before we begin to discuss the modelling of the heat transfer processes, I want to introduce an important relationship between energy and temperature. It tells us the amount of energy that is required to produce a given change in the temperature of a fixed amount of a substance, and we shall need it later on in this unit.

Think of a quantity of water being heated in an electric kettle. Electrical energy is being used up by the heating element (as the meter will testify). What happens to this energy? It goes to raise the temperature of the water and is therefore stored in the water as long as the water stays hot. While the water stays hot the energy originally supplied to it by the heating element is available for transfer elsewhere; to heat a cold bed by way of a hot water bottle, for example. The energy stored in a substance by virtue of its temperature is known as the **internal energy** of the substance; I shall denote it by the symbol E. This is another form of energy to add to those which you have already met: for example, the *kinetic energy* of a body (by virtue of its velocity) or its *potential energy* (by virtue of its position in, say, a gravitational field). The change in the internal energy of a mass m of water in raising its temperature from θ_1 to θ_2 is found to be

Unit 4

$$E_2 - E_1 = mc(\theta_2 - \theta_1) \tag{1}$$

where E_1 is the internal energy when the temperature is θ_1 and E_2 that when it is θ_2. The quantity c (which may, for the purposes of this unit, be considered constant) depends on the nature of the substance being heated, and is called the **specific energy capacity** or sometimes the **specific heat** of the substance.

Two points about Equation (1) are worth noting. The first is that this equation does not describe the *process* of heat transfer to the substance, but only the *result* of such a process. The second point is that this equation applies to any kind of substance—it does not have to be a liquid. It breaks down in complicated cases like mixtures of gases and liquids (e.g. when the water in the kettle begins to boil), but these cases are beyond the scope of this unit. For gases the relevant value of c is that which corresponds to heating at constant volume.

This brings me to the units we are going to use. As you know, we use the SI system of units in which energy is measured in joules (J) and mass in kilogrammes

See *Unit 4* and the *Handbook*.

(kg). We also need temperature, which is measured in degrees Celsius (°C) (commonly called centigrade). What then are the units of specific heat?

From Equation (1), we can put

$$c = \frac{E_2 - E_1}{m(\theta_2 - \theta_1)}.$$

Substituting the units of the terms $(E_2 - E_1)$, m and $(\theta_2 - \theta_1)$, we find that the units of c are $\frac{\mathrm{J}}{\mathrm{kg\,°C}}$ or $\mathrm{J\,kg^{-1}\,°C^{-1}}$ (in words, joules per kilogramme per degree Celsius).

Approximate values of c in these units for a number of substances are shown in Table 1.

Table 1

Substance		Specific heat $c/\mathrm{J\,kg^{-1}\,°C^{-1}}$
water		4200
alcohol		2500
copper		390
glass		630
air	heated at constant volume	715
hydrogen	heated at constant volume	10000

The oblique stroke means 'expressed in units of'.

There is one further quantity which we shall come across in the course of our work, and that is the rate at which energy is produced (e.g. by the heating element in the kettle). The unit of time in the SI system is the second, and so from what I have already said it is clear that the rate of energy production (or transmission or dissipation) is measured in joules per second ($\mathrm{J\,s^{-1}}$). This unit has a special name: it is the **watt** (W). Hence $1\,\mathrm{W} = 1\,\mathrm{J\,s^{-1}}$.

If, for example, the element in the kettle is rated at 3 kW, it will use electrical energy at the rate of 3000 joules per second, as long as it is switched on. (This rate of energy use is called the **power** of the heating element.)

Here are a few exercises to enable you to practise using the quantities and units I have mentioned in this section.

Exercise 1

Work out the change in the internal energy of 2 kg of water when its temperature is raised from 20° C to 80° C. Use Table 1.

Exercise 2

The water in Exercise 1 was heated in an electric kettle with an element rated at 2 kW. Assuming that all the electrical energy is used only in heating the water, estimate how long it will take to raise the water temperature from 20° C to 80° C.

Exercise 3

In Exercise 2 it is assumed that the electrical energy is used only to heat the water. What else would it be used for, in practice? Is the answer to Exercise 2 an underestimate or an overestimate?

[*Solutions to Exercises 1–3 on p. 27*]

Summary of Section 1

Heat transfer means energy transfer by heating. This kind of energy transfer requires the existence of a temperature difference, and the energy is transferred from regions or bodies of higher temperature to those of lower temperature. There are three modes of heat transfer: **conduction**, **radiation** and **convection**.

Part of the energy of a body depends on its temperature. This part is known as **internal energy**, and the change of internal energy due to a change in temperature $(\theta_2 - \theta_1)$ is $mc(\theta_2 - \theta_1)$, where:

m = mass of body,

c = **specific energy capacity** or **specific heat**.

2 Steady-state conduction in one dimension

2.0 Introduction

It has long been known that when different parts of a piece of material are at different temperatures, energy is transferred from the regions of higher temperature to those of lower temperature. The external walls of a house are a case in point. If the inside of the house is warmer than the air outside, then the inside surface of the wall will be warmer than the outside surface and energy will be transferred out of the house through the wall by conduction. It will then be transferred from the outside of the wall to the air, mainly by convection. In this section I want to concentrate on thermal conduction through solid walls.

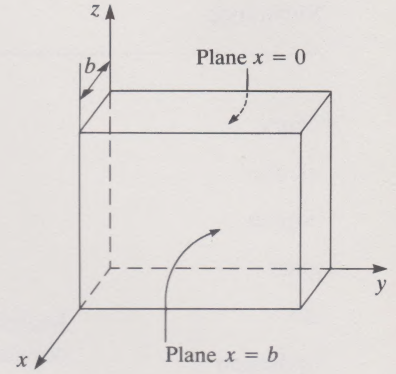


Figure 1

Here were are using three-dimensional Cartesian coordinates (x, y, z). They are explained in more detail in *Unit 14*.

The model I shall use depends on several simplifying assumptions:

(1) The wall can be represented as a uniform flat slab. In consequence we can choose coordinates so that one face of the slab is in the plane $x = 0$ (see Figure 1) and the other is in the plane $x = b$, where b is the thickness of the slab.

(2) The slab is in a **steady state**, which means that the temperature at any given point is independent of the time.

(3) The temperature in the slab is uniform in each plane x = constant. This implies that the temperature in the slab at any given time depends only on x and not on y or z. When the temperature depends on only one space coordinate in this way, we say that the temperature distribution is **one-dimensional**.

Assumptions (2) and (3) imply that the temperature in the slab, being independent of time t and the space coordinates y and z, depends only on the space coordinate x. Writing θ for the temperature we can say that θ is a function of x only, or in symbols

$$\theta = \theta(x) \qquad (0 \leq x \leq b).$$

The function $\theta(x)$ gives a complete description of the temperature distribution in the slab. The question now is how to find this distribution and how to find the rate of energy transfer across the slab.

2.1 Fourier's law

The model that we shall use for steady-state conduction in one dimension is based on theoretical work done by Joseph Fourier in the early nineteenth century which has since been amply confirmed by practical experience. Fourier's law gives quantitative expression to the idea that heat conduction transfers energy from hotter to cooler places, by postulating that the rate of energy transfer is determined by how rapidly the temperature varies with position.

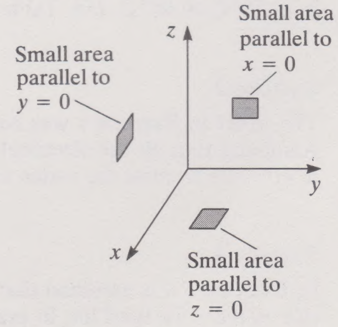


Figure 2

The law gives the rate of energy transfer by conduction across any small area, called an **area element**, within any block of material. For the one-dimensional problems considered in this unit, we need only consider area elements which are parallel to one of the coordinate planes. If the small area is parallel to one of the coordinate planes $y = 0$ or $z = 0$ (see Figure 2), then since the temperature does

not vary in the y and z directions, we may expect that no energy is transferred across it. On the other hand, for an area element parallel to the plane $x = 0$ (i.e. parallel to one of the faces of our slab), there will be transfer of energy.

We may expect the rate of heat transfer (q) across this area element to be proportional to its area, which may be denoted by a. The rate of heat transfer will also depend on how rapidly the temperature changes with position, that is to say $\frac{d\theta}{dx}$, which is called the **temperature gradient**. Fourier made the natural assumption that q is simply proportional to $\frac{d\theta}{dx}$. Combining these two proportionality relations, we are led to write

$$q \propto a\frac{d\theta}{dx}.$$

Before writing this as an equation, we should consider the sign of the constant of proportionality. If $\frac{d\theta}{dx}$ is positive, the temperature increases as x increases; and since heat flows from hot places to cold the heat transfer in this case will be from large values of x to small, i.e. in the negative x direction. The natural sign convention for q, however, is that a positive value of q corresponds to flow in the *positive* x direction. So $\frac{d\theta}{dx}$ and q have opposite signs. To allow for this, we write Fourier's law as an equation in the following form:

$$q = -\kappa a\frac{d\theta}{dx} \tag{1}$$

or

$$\frac{q}{a} = -\kappa\frac{d\theta}{dx}$$

where κ is a *positive* constant of proportionality.

The constant of proportionality κ in Equation (1) is called the **thermal conductivity** of the material through which conduction takes place, and different materials can have very different values of κ. Although Equation (1) assumes κ to be a constant, many materials are found in practice to have a value of κ which varies with temperature. For the purposes of this unit we can neglect this effect. Before we look at representative values of κ, let us just be clear about the SI units of thermal conductivity.

Question What are the SI units of thermal conductivity?

Answer From Equation (1) we have

$$\kappa = \frac{-q}{a\,d\theta/dx}.$$

Now the SI units of q are watts or W, the SI units of a are (metres)^2 or m^2, and the SI units of $\frac{d\theta}{dx}$ are $\frac{°\text{C}}{\text{metre}}$ or $\frac{°\text{C}}{\text{m}}$. The negative sign is irrelevant to the question of units.

So the units of κ are $\frac{\text{W}}{\text{m}^2\,°\text{C/m}} = \frac{\text{W}}{\text{m}\,°\text{C}}$

or $\quad \text{W}\,\text{m}^{-1}\,°\text{C}^{-1}$.

Table 2 shows some typical values of κ for various solid materials, all of them fairly common. Notice that the ratio of the largest value (for copper) to the smallest value (for cork board) is nearly 9000, and even for the two metals (copper and mild steel) the ratio is about 7.

Table 2

Material	Thermal conductivity $\kappa/\mathrm{W\,m^{-1}\,°C^{-1}}$
copper	380
mild steel	54
concrete	1.4
glass	1.0
brick	0.7
wood (teak)	0.17
asbestos board	0.16
cork board	0.043

2.2 Finding the temperature distribution

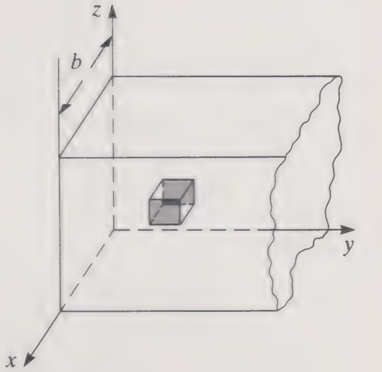


Figure 3

Before we can use Fourier's law to determine the temperature distribution $\theta(x)$, we need to know how q depends on x. To find this out, we now consider not an area element but a **volume element**, that is a small region inside the slab, bounded by faces parallel to the coordinate planes (see Figure 3).

We can apply the input–output principle (*Unit 3*)

$$\text{accumulation} = \text{input} - \text{output}$$

to this volume element. First of all, since we are assuming a steady state, the temperature of the volume element does not change and so its energy does not change. That is, there is no accumulation, so input and output must be equal. Our volume element has six faces, but there is no heat transfer across the four faces that are parallel to the coordinate planes $y = 0$ or $z = 0$ because the temperature does not vary in the y or z directions. The only heat transfer is across the two faces that are parallel to the $x = 0$ plane, which are shaded in Figure 3, and since input and output are equal, the heat transfer rates across these two faces must be equal. That is, for a steady state the values of q for the two opposite faces of a volume element are equal. With the geometry we are considering here, the areas of those faces are also equal, so $\dfrac{q}{a}$ is the same at the two ends of the volume element. It follows, by Fourier's law, that $\kappa\dfrac{d\theta}{dx}$ is the same at the two ends of the volume element.

The above argument applies to any volume element in the slab; for example, we could take one end of the volume element in the face $x = 0$ and the other end in some arbitrary plane $x = x_1$. Then the argument tells us that $\kappa\dfrac{d\theta}{dx}$ has the same value at $x = x_1$ as it does at $x = 0$. This is true for all x_1, and so we have shown that $\dfrac{d\theta}{dx}(x)$ is a constant.

Let us call this constant B; we then have

$$\frac{d\theta}{dx} = B.$$

This differential equation can be solved by direct integration to give

$$\theta = Bx + C$$

where C is another constant. This means that the graph of θ against x is a straight line.

To determine the constants B and C we need further information. For example, suppose we know the temperatures of the two faces of the slab to be θ_1 and θ_2, say. Then we have

$$\theta_1 = \theta(0) = C,$$
$$\theta_2 = \theta(b) = Bb + C$$

where b is the thickness of the slab measured in the x direction (Figure 3). Solving for B and C gives

$$B = \frac{\theta_2 - \theta_1}{b},$$
$$C = \theta_1.$$

Once we know the temperature distribution we can calculate the rate of heat transfer through the slab. By Fourier's law the heat transfer per unit area is

$$\begin{aligned}\frac{q}{a} &= -\kappa\frac{d\theta}{dx}\\ &= -\kappa B\\ &= -\kappa\left(\frac{\theta_2 - \theta_1}{b}\right).\end{aligned}$$

2.3 Conduction through a constant cross-sectional area

We are now ready to model steady-state conduction through a wall. I shall assume that the inside of the house is warmer than the outside and that energy transfer takes place only from the inside face of the wall to the outside face; I shall neglect any energy that passes to the rest of the structure via the edges of the wall. I shall also assume that the temperatures inside and outside are uniform and that the wall is made of uniform material like the slab I mentioned earlier. What we want our model to tell us is the rate at which energy passes through the wall.

In terms of the slab shown in Figure 3, the assumption that we can neglect any energy transfer through the edges of the wall implies that the temperature will not vary in the y or the z directions, and so we are justified in using the model described in Subsection 2.2. The total heat transfer rate will be the sum of the heat transfer rates of the elements, i.e. at any cross-section of the wall perpendicular to the x direction,

$$\begin{aligned}q_{\text{total}} &= \sum\left(-\kappa a\frac{d\theta}{dx}\right) \qquad \text{(where } \textstyle\sum \text{ stands for 'the sum of ...')}\\ &= -\kappa\frac{d\theta}{dx}\sum a\end{aligned}$$

since κ and $\dfrac{d\theta}{dx}$ are the same for all the elements.

It follows that

$$q_{\text{total}} = -\kappa A\frac{d\theta}{dx}$$

where A is the area of the wall at right angles to the x direction, so that $\sum a = A$.

Figure 4 shows the temperature variation in the x direction. Here, θ_1 is the temperature of the inside face and θ_2 is the temperature of the outside face, so that $\theta_1 > \theta_2$.

As before, $\dfrac{d\theta}{dx} = \dfrac{\theta_2 - \theta_1}{b}$ where b is the thickness of the wall, so that

$$\begin{aligned}q_{\text{total}} &= -\kappa A\left(\frac{\theta_2 - \theta_1}{b}\right)\\ &= \kappa A\left(\frac{\theta_1 - \theta_2}{b}\right). \qquad (2)\end{aligned}$$

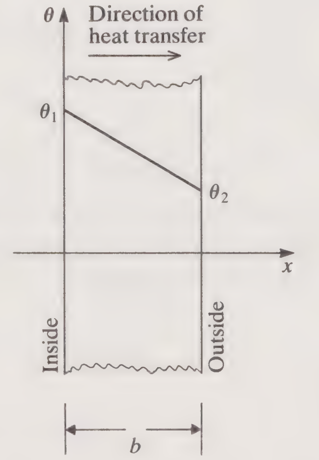


Figure 4

Equation (2) is the most commonly used version of the steady-state conduction equation. The following exercises are based on it so that you can see that it supplies the information we set out to obtain at the beginning of this subsection.

Exercise 1

The inside face of a brick wall 240 mm thick has a temperature of 18° C, and its outside face has a temperature of 10° C. Estimate the rate of heat transfer per m^2 of wall surface. (You will need to use Table 2.)

Exercise 2

What is the temperature gradient for the wall in Exercise 1? What is the temperature at a section of the wall 100 mm from the inner face?

[*Solutions to Exercises 1, 2 on p. 27*]

2.4 Conduction through a changing cross-section

I want to end this section by discussing the modelling of conduction through pipe walls—for example, central heating pipes which carry hot water to the radiators and are surrounded by cooler air. I shall again assume a steady state, as defined earlier, and also that heat transfer takes place *radially* (i.e. at each point in the pipe wall, the direction of heat transfer points away from the centre of the pipe). The main difference between this case and those we met in earlier pages is that this time the cross-sectional area through which energy is transferred is not constant. This becomes clear when we consider the cross-section of a pipe such as that shown in Figure 5. The pipe is of length l, and has inner radius r_1 and outer radius r_2 (so that the thickness of the pipe wall is $r_2 - r_1$).

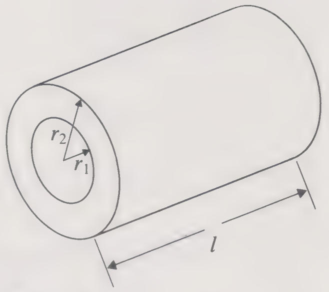


Figure 5

Consider a cylindrical surface of radius x inside the pipe wall (so that $r_1 \le x \le r_2$), as shown in Figures 6(i) and (ii).

Its area is $2\pi x l$. By symmetry, the temperature at all points of this surface will be the same. It follows that the steady-state temperature θ in the pipe wall will be a function only of x and the temperature gradient will be $\dfrac{d\theta}{dx}$. If we consider a small part of the curved surface, it will behave just as if it were flat, and so the rate of heat transfer per unit area is, as in the case of the flat slab, equal to $-\kappa\dfrac{d\theta}{dx}$. Hence if q is the steady-state heat transfer rate through the pipe wall and A is the area of the cylindrical surface,

$$\frac{q}{A} = -\kappa\frac{d\theta}{dx}$$

or

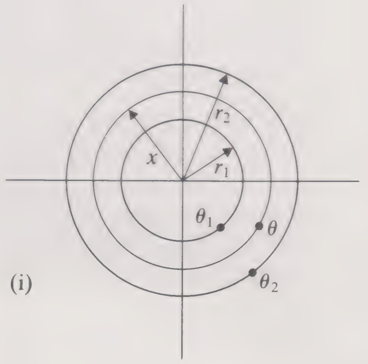
$$q = -\kappa A\frac{d\theta}{dx}$$

where A is not constant, but a function of x. In fact, as we have seen, $A = 2\pi x l$, and so

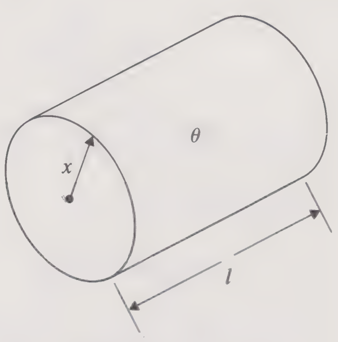
$$q = (-\kappa) \times (2\pi x l) \times \left(\frac{d\theta}{dx}\right).$$

Hence

$$\frac{d\theta}{dx} = \frac{-q}{2\pi\kappa x l}. \tag{3}$$



(i)



(ii) Figure 6

Unit 2, Subsection 3.1

In the steady-state, q is a constant. Thus the right-hand side of Equation (3) is a known function of x, and this equation can be solved by direct integration to give the general solution

$$\theta = \int \frac{-q}{2\pi\kappa x l}\,dx$$

$$= -\frac{q}{2\pi\kappa l}\log_e x + C \tag{4}$$

where C is a constant of integration.

Example

Sketch the temperature variation along a radius of a pipe carrying hot liquid. Will the temperature halfway between the inside and outside surfaces be greater or smaller than it would be for a flat wall of the same material and thickness with the same inside and outside temperatures?

Solution

We use the notation of Figure 6(i). The inside temperature θ_1 will be the greatest. The gradient of the graph of θ against x will be negative at all points but flattening out as x increases. This follows from Equation (3). The required graph is sketched in Figure 7 (full line). The graph for a flat wall is shown dotted. Clearly the half-way temperature is *less* for the pipe than for the flat wall.

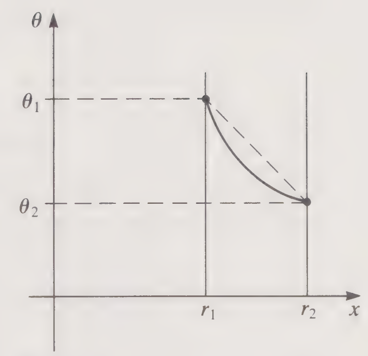


Figure 7

By substituting into Equation (4) the values appropriate to the inside and outside curved surfaces of the pipe, we get

Remember that q is constant and has the same value at every radius.

$$\theta_1 = -\frac{q}{2\pi\kappa l}\log_e r_1 + C$$

and

$$\theta_2 = -\frac{q}{2\pi\kappa l}\log_e r_2 + C.$$

Subtracting the second from the first, we have

$$\begin{aligned}\theta_1 - \theta_2 &= -\frac{q}{2\pi\kappa l}(\log_e r_1 - \log_e r_2)\\ &= \frac{q}{2\pi\kappa l}(\log_e r_2 - \log_e r_1)\\ &= \frac{q}{2\pi\kappa l}\log_e\left(\frac{r_2}{r_1}\right)\end{aligned}$$

so

$$q = \frac{2\pi\kappa l(\theta_1 - \theta_2)}{\log_e (r_2/r_1)}. \tag{5}$$

Exercise 3

A glass pipe has an inside diameter of 50 mm and an outside diameter of 80 mm. The inside surface of the pipe is at a temperature of 90° C and the outside surface is at 50° C. Estimate the rate at which energy is conducted through the wall of the pipe per metre length of pipe, in the steady state.

Exercise 4

For the pipe specified in Exercise 3 what is the temperature in the pipe wall at a radius of 32 mm?

[*Solutions to Exercises 3 and 4 on p. 27*]

Summary of Section 2

Fourier's law for steady-state conduction in one dimension is

$$q = -\kappa a\frac{d\theta}{dx}$$

where: q = heat transfer rate in the positive x direction,
κ = **thermal conductivity**,
a = area at right angles to the direction of heat transfer,
θ = temperature,
x = distance measured in direction of temperature variation.

$\dfrac{d\theta}{dx}$ is known as the **temperature gradient**.

For steady-state conduction through a uniform slab of constant cross-sectional area A, Fourier's law reduces to

$$q = \kappa A \frac{(\theta_1 - \theta_2)}{b}$$

where $(\theta_1 - \theta_2)$ is the temperature drop across thickness b of the material.

For steady-state conduction through a pipe of circular cross-section of internal radius r_1, external radius r_2 and length l, Fourier's law leads to

$$q = \frac{2\pi\kappa l(\theta_1 - \theta_2)}{\log_e (r_2/r_1)}$$

where $(\theta_1 - \theta_2)$ is the temperature drop across the pipe wall.

3 Television programme notes

TV 12

Now watch the television programme: 'One-dimensional steady state heat transfer'.

Read Section 3 after viewing the programme. (This section is intended to be read after watching the television programme, but you should read it and work through the numerical parts even if you missed the programme, because it contains new assessable material.)

The main purpose of the television programme is to investigate the practical usefulness of the simple models of heat transfer which are derived in the unit, especially that for conduction. Dr Audrey Stuckes is studying the thermal insulation properties of building materials. She is looking at the performance of fibrous roof insulation materials for the Building Research Establishment and at the properties of polymeric materials in wall structures for the Polymer Engineering Directorate. The experiments comprise the careful measurement, under realistic conditions, of temperatures and heat transfer rates. Although conditions in practice are rather more complicated, it turns out that the heat transfer in buildings can be adequately modelled by one-dimensional steady-state models. Indeed, if the models are to be really useful to builders and architects, they must be simple.

In order to remind you of the simple model for conduction and to introduce the measuring devices used in the work described so far, the programme includes a simulation of one-dimensional heat conduction along a metal bar which has a heater at one end and heat flow meter at the other. (We have used a simulation because we wanted to remind you of the fact that when the heater is first switched on the temperature at any point in the bar will change with time but will eventually settle down to a steady value, and the time required to do this in an actual experiment would have been excessive.)

The main point, made with the help of animated diagrams, is to illustrate what is meant by 'one-dimensional steady-state heat transfer by conduction'.

Exercise 1 (revision)

The metal bar mentioned above is 12 mm in diameter and 200 mm long. The steady-state temperature difference between its ends was shown as 10° C and the heat flow meter showed a reading of about 1.12 W. Estimate the thermal conductivity of the material of the bar.

[*Solution on p. 27*]

Although conduction is the most significant mode of heat transfer in building materials, convection and radiation also play their part. Yet the total heat transfer rate is, in practice, usually modelled by an even simpler expression than that for conduction. This is done by using a quantity called the **U value**, which is a kind of overall heat transfer coefficient. It can be, and is, used in cases where heat transfer occurs across more than one layer of material (as for example, in a cavity wall). The U value in such a case is worked out by dividing the heat transfer rate per unit area by the total temperature difference across the material in question, and it

can take account not only of conduction through solid materials but also of convection (as in the heat transfer between a wall and the air on either side of it) and radiation effects. Here is an example.

Example 1

The air temperatures on either side of a cavity wall are 21° C and 0° C respectively. It is found that the total heat transfer rate per square metre of wall is 30 W. What is the U value for the wall?

Solution

$$U \text{ value} = \frac{\text{heat transfer rate per unit area}}{\text{total temperature difference}}$$

$$= \tfrac{30}{21} = 1.43 \ \mathrm{W\,m^{-2}\,{}^\circ C^{-1}}.$$

I shall have more to say about U values in Section 4. For the moment it is enough to know that the U value is a measure of the ease with which heat transfer can take place and that at present, because of the need to conserve energy, there is a premium on methods of building construction which result in low U values.

The programme also features demonstrations (not simulations) of convection and radiation. Convection is shown by the movement of particles in a liquid which is heated at one point. Radiation from an electric heater, via a reflector, is used to ignite a match. A very important fact about radiation is mentioned, namely that the rate of energy emission by radiation depends on the fourth power of the **absolute temperature** of the radiating body. This needs a little more explanation, as follows.

Radiated energy is emitted and absorbed by any given body all the time. In this unit we deal only with cases where the absorbed radiation is very much less than the emitted radiation (for example, an electric lamp) and where we can say with little error that the energy emitted by radiation equals the net heat transfer rate, q. Now in such a case,

$$q = \mu\theta_A^4$$

where μ is a constant and θ_A is the **absolute temperature**, defined as follows:

$$\theta_A = 273 + \theta$$

where θ is temperature in degrees Celsius. This definition implies that the zero of the absolute temperature scale is at $-273°$ C. This point represents the lower limit of attainable temperature, and so the absolute scale of temperature is not based on the properties of a particular substance (in the way that 0° C refers to the freezing of water, for example).

The SI unit for absolute temperature is the Kelvin, represented by the symbol K as in the statement '0° C is equivalent to +273 K'. Note: (i) that one writes 'K' and *not* 'degrees K'; (ii) that the temperature *intervals* are the same on the Kelvin scale as on the °C scale. (For example, the temperature *difference* between 240 K and 230 K is the same as that between 240° C and 230° C.) This is why the units of, for example, thermal conductivity are often quoted as $\mathrm{W\,m^{-1}\,K^{-1}}$, which is exactly equivalent to $\mathrm{W\,m^{-1}\,{}^\circ C^{-1}}$ since temperature differences are relevant here rather than particular values of temperature.

It is also worth remembering that, as I said in Section 1, heat transfer by radiation can take place across empty space.

Exercise 2

Write down the absolute temperature, on the Kelvin scale, which corresponds to each of the following:

(i) $-250°$ C, (ii) 3° C.

Exercise 3

A vacuum lamp is rated at 40 W. Write down an expression for the heat transfer rate from the lamp as a function of the filament temperature θ °C and estimate the value of μ if $\theta = 2000$.

[*Solutions to Exercises 2 and 3 on p. 27*]

Summary of Section 3

The television programme includes demonstrations of conduction, convection and radiation, and of research work into the thermal insulation properties of building materials carried out for the Building Research Establishment and the Polymer Engineering Directorate.

The heat transfer through walls and roofs is in practice often represented by simple, steady-state, one-dimensional models. In particular the ***U* value** is in common use as an index of how readily heat transfer can take place. The U value is defined as follows:

$$U \text{ value} = \frac{\text{heat transfer rate}}{\text{overall temperature difference}}.$$

New techniques and components are being developed to satisfy the demand for very low U values in buildings.

The thermal energy emitted by radiation from a body depends on the fourth power of the absolute temperature of the body, i.e.

thermal energy emitted by radiation $= \mu\theta_A^4$, where:

μ is a constant,

$\theta_A = 273 + \theta =$ absolute temperature,

$\theta =$ temperature in °C.

4 Convection and insulation

4.1 A simple model of convection

So far, we have looked at conduction through solid walls without paying much attention to the way in which the energy is transmitted to and from the wall. This is what I want to consider now. In the case of the house wall, energy is transferred from the warm air in the room, through the wall, to the cool air outside. There are therefore two regions where there is heat transfer between a fluid (air) and a solid (the wall). In the case of the pipe, the energy is transferred from the hot water inside to the cool air outside, and again there are two fluid–solid boundaries (water–pipe and pipe–air) across which heat transfer takes place. In either case, since we are dealing with a fluid, convection will play a dominant part in the heat transfer process associated with the boundary.

Considered in detail, convection turns out to be a rather complicated matter, not at all easy to analyse or to model. As you would expect from what I said in Section 1, fluid mechanics plays a very important part in any thorough consideration of convection. Fortunately, in spite of the fact that fluid mechanics is outside the scope of this course, it is possible to get some quite useful results based on a very simple description of the main features of convection, together with some of the practical experience that has been accumulated over the years.

Figure 1 shows a solid–fluid boundary, and I shall assume that heat transfer is taking place from the solid (say a house wall) to the fluid (atmospheric air). As we know, this necessarily means that the temperature of the face of the wall, where it touches the air, must be higher than the air temperature at some distance from the house. The parts of the air near the wall are heated by conduction and radiation before they move away to join the main body of the air, being replaced by cooler air which in turn is heated. The result is that the air very near the wall has a temperature higher than that of the main body of the air, which is assumed to be at a uniform temperature. So the temperature drop between the wall and the main body of the air, which enables heat transfer to take place, is concentrated in a thin layer of the air quite near the solid boundary. This is sketched in Figure 1, where θ_s represents the temperature of the wall at its boundary and θ_f represents the temperature of the main body of the air.

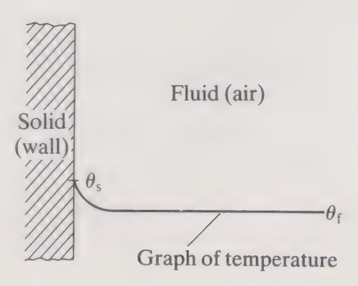


Figure 1

The same general argument applies when the direction of the heat transfer is from the fluid to the solid; the temperature drop from fluid to solid is again concentrated in a thin layer of fluid immediately next to the solid boundary. In this layer the fluid is giving up some of its energy to the wall, mainly by conduction. Figure 2 shows part of the cross-section of a solid house wall, together with a sketch of the temperature variation from the inside air at temperature θ_{in} to the outside air at temperature θ_{out}. We know how the temperature drop $(\theta_1 - \theta_2)$ across the wall is related to the heat transfer rate—but what about $(\theta_2 - \theta_{out})$? It is reasonable to assume that the nature of the solid surfaces and the nature of the fluid will have something to do with it. We also need to know whether the fluid near the wall is stationary or moving. (Inside the room the air is presumably stationary or at most moving very slowly; outside the house there may be strong winds.)

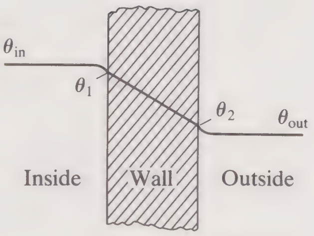


Figure 2

The model for convection that is commonly used in practice by chemical engineers, architects and heating engineers is very simple. It is as follows:

$$q = hA\,\Delta\theta \tag{1}$$

$\Delta\theta$ is to be treated as a single symbol. It does **not** mean $\Delta \times \theta$.

where A = area of the solid–fluid boundary,
$\Delta\theta$ = temperature difference between solid and fluid,
q = rate of heat transfer,

and h is a quantity called the **convective heat transfer coefficient.**

Equation (1) looks similar to the equation we used to model conduction through a slab, with h substituted for $\frac{\kappa}{b}$. In fact, h is a different sort of coefficient because it takes into account not only the nature of the solid surface but also the nature of the adjacent fluid, as well as the relative velocity between fluid and solid surface.

Question

What are the SI units of the convective heat transfer coefficient h?

Answer

$$h = \frac{q}{A\,\Delta\theta}$$

so that the units are $\frac{\mathrm{W}}{\mathrm{m}^2\,{}^\circ\mathrm{C}}$ or $\mathrm{W\,m^{-2}\,{}^\circ C^{-1}}$.

Question

In the case of the house wall in Figure 2, will the value of h be greater for the inside surface or the outside surface?

Answer

The greater movement of the air outside will speed up the process of convection and hence the heat transfer rate. Consequently h will be greater for the outside surface than the inside surface.

The actual value of h to be used in Equation (1) is based on measurements of heat transfer rates, temperatures and air speeds. Over the years a great deal of such information has built up and has been proved in practice, so that there are accepted values of h for many purposes. An idea of the order of magnitude of h, and its variability with different materials, is given in Table 3.

Table 3

Process	Type of boundary	$h/\mathrm{W\,m^{-2}\,{}^\circ C^{-1}}$
free convection	solid–gas solid–liquid	0.5–1 000 100 –5 000
forced convection	solid–gas solid–liquid	10 –1 000 100 –10 000

We can now go back to Figure 2 and write down an expression for the heat transfer right across from the warm air inside the room to the colder air outside. In the steady state, the rate of heat transfer will be the same at all points so that, if A is the effective area and b is the thickness of the wall, we can write

$$\frac{q}{A} = h_{\text{in}}(\theta_{\text{in}} - \theta_1) = \frac{\kappa}{b}(\theta_1 - \theta_2) = h_{\text{out}}(\theta_2 - \theta_{\text{out}})$$

where h_{in} and h_{out} are the convective heat transfer coefficients at the inside and outside surface respectively.

Example 1

By using the three heat transfer equations for Figure 2, show that

$$q = A(\theta_{\text{in}} - \theta_{\text{out}})\left\{\frac{1}{h_{\text{in}}} + \frac{b}{\kappa} + \frac{1}{h_{\text{out}}}\right\}^{-1}.$$

Solution

From the three equations, we have:

$$\theta_{\text{in}} - \theta_1 = \frac{q}{Ah_{\text{in}}},$$

$$\theta_1 - \theta_2 = \frac{qb}{A\kappa},$$

$$\theta_2 - \theta_{\text{out}} = \frac{q}{Ah_{\text{out}}}.$$

Adding these equations, we get

$$\theta_{\text{in}} - \theta_{\text{out}} = \frac{q}{A}\left\{\frac{1}{h_{\text{in}}} + \frac{b}{\kappa} + \frac{1}{h_{\text{out}}}\right\}.$$

Hence

$$q = A(\theta_{\text{in}} - \theta_{\text{out}})\left\{\frac{1}{h_{\text{in}}} + \frac{b}{\kappa} + \frac{1}{h_{\text{out}}}\right\}^{-1}$$

as required.

The expression which you were asked to derive in Example 1 gives the heat transfer rate in terms of the overall temperature drop. The quantity

$$\left\{\frac{1}{h_{\text{in}}} + \frac{b}{\kappa} + \frac{1}{h_{\text{out}}}\right\}^{-1}$$

is usually referred to as the ***U* value** for the wall. The U value is the overall heat transfer coefficient for the wall and the surface effects combined. So the heat transfer rate can be written

You met U values in the television programme.

$$q = A\,U(\theta_{\text{in}} - \theta_{\text{out}}) \tag{2}$$

where

$$U = \left\{\frac{1}{h_{\text{in}}} + \frac{b}{\kappa} + \frac{1}{h_{\text{out}}}\right\}^{-1}.$$

In the technical literature, $\dfrac{1}{U} = \dfrac{1}{h_{\text{in}}} + \dfrac{b}{\kappa} + \dfrac{1}{h_{\text{out}}}$ is usually referred to as the **total thermal resistance** of the wall.

Exercise 1

A brick wall of a house, 0.24 m thick, has a thermal conductivity of 0.7 W m^{-1} °C^{-1}. The air temperature on one side of the wall is 21° C and on the other side the temperature is 0° C. The convective heat transfer coefficients are 8 W m^{-2} °C^{-1} for the high temperature side and 19 W m^{-2} °C^{-1} for the low temperature side. Calculate the U value for the wall.

Exercise 2

For the wall specified in Exercise 1, estimate the heat transfer rate per square metre of wall. What is the thermal resistance of the wall?

Exercise 3

If the resistance of the air film next to each side of the wall were to be neglected in Exercises 1 and 2 (i.e. if it were to be assumed that $\frac{1}{h_{in}} = \frac{1}{h_{out}} = 0$), calculate the resulting percentage error in the heat transfer rate.

[*Solutions to Exercises 1–3 on p. 28*]

Because of the varying area across which heat transfer takes place in pipes, U-values in the strict sense are not conveniently calculated, but an equivalent procedure is the subject of the following example.

Example 2

Derive an expression for the heat transfer rate in terms of the overall temperature difference for a pipe of circular cross-section carrying hot water and surrounded by cooler air.

Sketch a graph showing the temperature variation between the hot water and the air.

Solution

I shall use the notation of Subsection 2.4 (i.e. inner radius r_1, outer radius r_2, corresponding temperatures θ_1 and θ_2). In addition I shall call the temperature of the hot water θ_{in}, that of the cool air θ_{out}, and the convective heat transfer coefficients at the inside and outside surfaces h_{in} and h_{out} respectively.

Then the equations for heat transfer by convection at the inner and outer surfaces give

$$q = 2\pi r_1 l h_{in}(\theta_{in} - \theta_1) = 2\pi r_2 l h_{out}(\theta_2 - \theta_{out}),$$

while the equation for heat transfer by conduction through the pipe wall is given by Equation (5) of Subsection 2.4 as

$$q = \frac{2\pi\kappa l(\theta_1 - \theta_2)}{\log_e(r_2/r_1)}.$$

Hence

$$\theta_{in} - \theta_1 = \frac{q}{2\pi r_1 l h_{in}},$$

$$\theta_2 - \theta_{out} = \frac{q}{2\pi r_2 l h_{out}},$$

$$\theta_1 - \theta_2 = \frac{q\log_e(r_2/r_1)}{2\pi\kappa l}.$$

Adding,

$$\theta_{in} - \theta_{out} = \frac{q}{2\pi l}\left\{\frac{1}{r_1 h_{in}} + \frac{1}{\kappa}\log_e(r_2/r_1) + \frac{1}{r_2 h_{out}}\right\}.$$

Thus,

$$q = 2\pi l(\theta_{in} - \theta_{out})\left\{\frac{1}{r_1 h_{in}} + \frac{1}{\kappa}\log_e(r_2/r_1) + \frac{1}{r_2 h_{out}}\right\}^{-1}.$$

The required sketch is Figure 3.

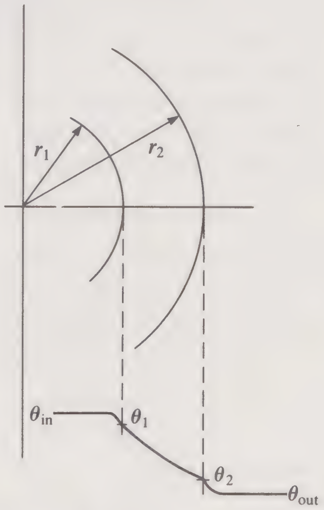


Figure 3

4.2 Design for cooling

Throughout most of this unit so far I have used examples in which heat transfer has, by implication, been presented as an undesirable phenomenon. Indeed the title of this section includes the word 'insulation', which simply means the cutting down of heat transfer rates. This is a very topical subject at a time when economy in the use of energy is, or should be, the order of the day, and I shall attempt to give it its due in the next subsection. In the meantime, however, it seems appropriate to remind ourselves that there are many occasions when heat transfer is desirable and ingenuity is properly exercised in fostering it.

One such example is an air-cooled internal combustion engine, such as most motor-cycle engines. You must have noticed what the external cylinder wall of such an engine looks like—it has deep corrugations to increase the area of contact with the atmospheric air which acts as coolant. The same idea of an 'extended surface' is used in refrigerators and in industrial heat exchangers (which are heating or cooling devices) where, for example, pipes are fitted with fins as in Figure 4.

Figure 4

I want to take a look at such a finned pipe, in order to model the effect of the fins. I shall assume that the pipe is carrying a hot liquid which is to be cooled by the air surrounding the tube. I propose to model a single fin on the assumption that its action is not interfered with by the other fins: in other words that each fin transfers energy to air at the ambient atmospheric temperature (not heated by the other fins).

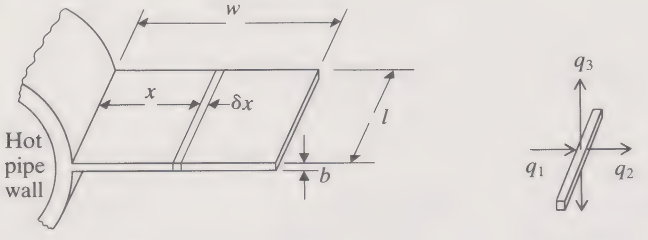


Figure 5

A single such fin is shown in Figure 5. Since its width is considerably greater than its thickness, I shall, for simplicity, neglect the heat transfer from its edges and model only the heat transfer from its top and bottom faces. This is a pessimistic assumption as far as the performance of the fin is concerned. We shall look at a small but finite length l of pipe and fin. Also, as before, we shall assume a steady state in which the temperature at any point does not change with time, and also that the temperature at any distance x from the pipe is uniform over the whole cross-section of the fin.

We begin by considering the energy input to and output from a small element of the fin having width δx at distance x from the pipe wall; that is to say the energy transferred to and from that element, which is shown in Figure 5(i). Figure 5(ii) is a picture of the element in isolation. q_1 is the rate of energy transfer by conduction *to* the element from the part of the fin on the left of the element. q_2 is the rate of energy transfer by conduction *from* the element to the rest of the fin on the right of the element. q_3 is the rate of energy transfer (both sides) *from* the element to the atmosphere. Now if the temperature of the element is to remain constant, then during any interval of time, the amount of energy given to the element must equal the amount of energy removed from it. I mentioned this when I discussed the steady state in Subsection 2.2. Any net gain or loss of energy would result in a change of internal energy and hence in a temperature change. Thus we can write

$$q_1 = q_2 + q_3. \tag{3}$$

The next step is to try to express all the quantities in Equation (3) in terms of the dimensions and physical properties of the fin.

For example, Fourier's law tells us that

$$q = -\kappa A \frac{d\theta}{dx}$$

where $A = lb$,

$\kappa =$ thermal conductivity of fin material,

$\theta(x) =$ temperature of fin at distance x from pipe wall.

It follows that

$$q_1 = -\kappa lb \frac{d\theta}{dx}(x)$$

and

$$q_2 = -\kappa lb \frac{d\theta}{dx}(x + \delta x).$$

As for q_3, I shall assume that all the values of θ are low enough to make radiation inconsiderable and I shall treat it as the convective heat transfer rate from the element to the atmosphere. The area from which this takes place is $2l\delta x$ (top and bottom faces with the edge neglected). So if the atmospheric temperature is θ_a and the relevant convective heat transfer coefficient is h, then

$$q_3 \simeq h \times 2l\delta x \times (\theta - \theta_a). \tag{4}$$

This equation is approximate because we have neglected (among other things) the variation in $\theta(x)$ across the element. The relative error in this approximation reduces as δx is made smaller.

Putting these expressions for q_1, q_2 and q_3 into the 'input–output' formula (3) (rearranged into the form $q_3 = q_1 - q_2$) gives

$$2h(\theta - \theta_a)\delta x \simeq \kappa lb\left\{\frac{d\theta}{dx}(x + \delta x) - \frac{d\theta}{dx}(x)\right\}. \tag{5}$$

Since δx is small, we can approximate the expression in square brackets by using the Taylor (tangent) approximation about x for the function $\dfrac{d\theta}{dx}$. This approximation is

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$$\frac{d\theta}{dx}(x + \delta x) = \frac{d\theta}{dx}(x) + \delta x\frac{d^2\theta}{dx^2}(x) + \cdots$$

and Equation (5) therefore simplifies to

$$2lh(\theta - \theta_a)\delta x \simeq \kappa lb\delta x\frac{d^2\theta}{dx^2} + \cdots$$

Dividing by $l\delta x$ gives

$$2h(\theta - \theta_a) \simeq \kappa b\frac{d^2\theta}{dx^2} + \cdots \tag{6}$$

The sign '$\simeq$' and the '$+ \ldots$' refer to approximations which become better and better the smaller δx becomes. We can therefore make Equation (6) as accurate as we please by making δx small enough. Taking the limit of vanishingly small δx, we may therefore dispense with the approximation signs and write

$$2h(\theta - \theta_a) = \kappa b\frac{d^2\theta}{dx^2} \tag{7}$$

although of course even this equation is not an exact model of the fin problem because of other approximations, already discussed, which do not depend on the size of δx.

Equation (7) is a linear second-order differential equation and can be solved using methods given earlier in this course. For conciseness we first define

See *Unit 6*

$$\lambda = \sqrt{\frac{2h}{\kappa b}}$$

so that Equation (7) becomes

$$\frac{d^2\theta}{dx^2} - \lambda^2\theta = -\lambda^2\theta_a. \tag{8}$$

Question

What is the general solution of Equation (8)?

Answer

The general solution of

$$\frac{d^2\theta}{dx^2} - \lambda^2\theta = -\lambda^2\theta_a$$

is (as we saw in *Unit 6*) the sum of two functions:

(i) any one *particular solution*, and

(ii) the *complementary function*, which is the general solution of

$$\frac{d^2\theta}{dx^2} - \lambda^2\theta = 0.$$

Hence the complementary function is

$$Fe^{\lambda x} + Ge^{-\lambda x},$$

where F and G are constants. It is easy to see that a particular solution of (8) is

$$\theta = \theta_a.$$

Hence the general solution of (8) is

$$\theta = Fe^{\lambda x} + Ge^{-\lambda x} + \theta_a. \qquad (9)$$

Question

Briefly, how do we find F, G and θ_a?

Answer

θ_a has already been defined as the atmospheric temperature, which can be found by measurement. To find F and G we need two boundary conditions, which when substituted in (9) will give two simultaneous equations to solve for F and G.

One of the two boundary conditions we need arises directly from the way in which we derived Equation (3), from Figure 5(ii). Clearly, when $x = w$, $q_1 = 0$ because there is no part of the fin beyond $x = w$ which could receive energy by conduction. Hence when $x = w$,

$$q_1 = -\kappa A\frac{d\theta}{dx} = 0,$$

and since κ and A are non-zero it follows that $\dfrac{d\theta}{dx} = 0$ when $x = w$.

Here, then, is one of our boundary conditions. The most convenient way to obtain the other one is to specify the temperature of the outside of the pipe wall, where the fin is attached. This can be measured without much difficulty. We are dealing with the steady state and so this temperature, like the others, will be independent of time. We therefore state our second boundary condition as

$$\theta = \theta_0 \qquad \text{when } x = 0 \qquad (\theta_0 \text{ constant}).$$

Exercise 4

By using the boundary conditions stated above, derive F and G in Equation (9) in terms of θ, λ and w, and show that

$$\theta - \theta_a = \frac{T}{1 + e^{2\lambda w}}\{e^{\lambda x} + e^{(2\lambda w - \lambda x)}\}$$

where $T = \theta_0 - \theta_a$.

Exercise 5

The following data apply to a finned copper tube:

$w = 50\,\text{mm}$	$h = 10\ \text{W}\,\text{m}^{-2}\,°\text{C}^{-1}$
$b = 2\,\text{mm}$	$\theta_0 = 150°\,\text{C}$
$\kappa = 380\ \text{W}\,\text{m}^{-1}\,°\text{C}^{-1}$	$\theta_a = 20°\,\text{C}.$

(i) Estimate the temperature midway along the fin.

(ii) Treating the temperature midway along the fin as the average temperature over the whole length of the fin, estimate the heat transfer rate from the whole fin per metre of axial length.

[*Solutions to Exercises 4 and 5 on p. 28*]

4.3 Insulation

By insulation we mean a layer (or several layers) of material which does not easily transfer energy and which is deliberately introduced in order to reduce the rate of heat transfer. Very often insulation is used to save fuel. Lagging of pipes is one example: pipes are lagged by wrapping them around with layers of a material of low thermal conductivity. Still air is a very good insulator; this is why 'cavity walls' are successful in house-building: they consist of two layers of bricks separated by an air space. Double glazing with its air space between two glass panels works on much the same principle. Perhaps the commonest and best known example of insulation is provided by the clothes we wear. Their main function, at least in the British climate, is to maintain a layer of still air between the skin and the atmosphere in order to keep the skin temperature at a comfortable level. In very hot countries white clothes may also act as reflectors of radiation.

However, things are not quite as simple as that. It is true that still air is a poor conductor, but as we have seen, air when heated may not remain still and is then capable of heat transfer by convection. This is why the insulating properties of, for example, double glazing do not necessarily improve as the thickness of the air gap is increased, as would be the case if conduction were the only heat transfer mechanism. Instead the variation of heat transfer by conduction and convection combined is as shown in Figure 6 in terms of a **combined heat transfer coefficient** h_c. This one coefficient takes into account both conduction and convection in the air gap. The resistance of the air gap can therefore be expressed in the one term $\frac{1}{h_c}$ instead of the three that would be necessary if conduction and convection had to be treated separately. Clearly, as the air space is made thicker there is an improvement up to a point, after which it actually gets worse because of the effect of convection. The optimum thickness depends on the temperature difference across the air—for a temperature drop of about 33° C, for example, the best thickness for the air gap is about 15 mm. (I am assuming that the double glazing is fitted in order to provide thermal insulation only—its use for sound insulation is beyond the scope of this unit.) This effect, of convection preventing us from getting the full benefit of reduced conductivity, is found also in cavity walls. This is the reason why nowadays the cavity is often filled with plastic foam or granules. In the foam, and between the granules, there are many small air spaces, separated from one another so that there is no convection from one to the other.

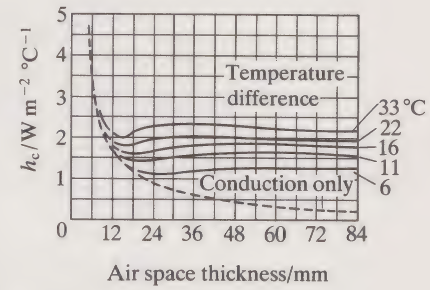


Figure 6

The lagging of a pipe can also have its snags. One effect of winding on the insulating material is to increase the surface area in contact with the surroundings, and this tends to increase the heat transfer rate due to convection, although of course the surface temperature is reduced by the insulation. This point is taken up again in Problem 3 of Section 5.

The following calculations are intended to illustrate some of the points I have made in this subsection, particularly the use of the combined convection–conduction coefficient.

Example 3

Assuming that heating a house costs 16p per **therm** (1 therm = 1.055×10^8 J), work out the cost of the energy which passes through 1 m^2 of window in one hour when the inside temperature is 18° C and the outside temperature 0° C:

(i) for a single glass pane,

(ii) for double glazing.

Assume the following data.

Measurement data: glass panes are 6 mm thick, with a 12 mm air space for double glazing.

Heat transfer data: $\kappa = 1$ for glass, units W m^{-1} °C^{-1}

$h = 10$ for the inside face, 20 for the outside face. } units W m^{-2} °C^{-1}

Solution

(i) For a single glass pane (Figure 7), the energy transmitted per second per m^2 is given by three equations:

$$q = 10(18 - \theta_1), \text{ i.e. } 18 - \theta_1 = \frac{q}{10} \text{ (inside convective heat transfer);}$$

$$q = 1\left(\frac{\theta_1 - \theta_2}{0.006}\right), \text{ i.e. } \theta_1 - \theta_2 = 0.006q \text{ (heat conduction through glass);}$$

$$q = 20(\theta_2 - 0), \text{ i.e. } \theta_2 = \frac{q}{20} \text{ (outside convective heat transfer).}$$

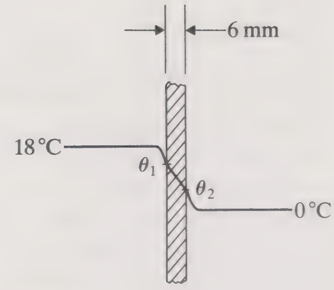


Figure 7

Adding these three equations:

$$18 = q\left(\frac{1}{10} + 0.006 + \frac{1}{20}\right) = 0.156q,$$

so that

$$q = \frac{18}{0.156}.$$

Thus in one hour, the energy transmitted through 1 m^2 of window is

$$\frac{18}{0.156} \times 3600\,\text{J} = 415\,385\,\text{J}.$$

$$\text{Cost} = 16 \times 415\,385 \times \frac{10^{-8}}{1.055} = 0.0630 \text{ pence.}$$

(ii) For double glazing (Figure 8), the energy transmitted per second per m^2 is given by five equations:

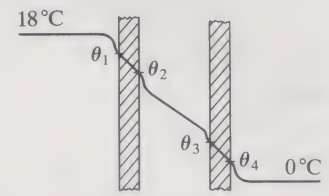


Figure 8

$$q = 10(18 - \theta_1), \text{ i.e. } 18 - \theta_1 = \frac{q}{10} \text{ (inside convective heat transfer);}$$

$$q = 1\left(\frac{\theta_1 - \theta_2}{0.006}\right), \text{ i.e. } \theta_1 - \theta_2 = 0.006q \text{ (heat conduction through inner pane);}$$

$$q = h_c(\theta_2 - \theta_3), \text{ i.e. } \theta_2 - \theta_3 = \frac{q}{h_c} \text{ (heat transfer through air space);}$$

$$q = 1\left(\frac{\theta_3 - \theta_4}{0.006}\right), \text{ i.e. } \theta_3 - \theta_4 = 0.006q \text{ (heat conduction through outer pane);}$$

$$q = 20(\theta_4 - 0), \text{ i.e. } \theta_4 = \frac{q}{20} \text{ (outside convective heat transfer).}$$

Adding:

$$18 = q\left(\frac{1}{10} + 0.006 + \frac{1}{h_c} + 0.006 + \frac{1}{20}\right),$$

$$q = 18\left(\frac{1}{10} + 0.006 + \frac{1}{h_c} + 0.006 + \frac{1}{20}\right)^{-1}.$$

In order to evaluate this we need to find the value of h_c from Figure 6. This, in turn, requires a knowledge of the temperature drop in the airspace, i.e. $\theta_2 - \theta_3$. We can get an estimate of this by using a process of successive approximations.

As a first approximation assume that the temperature difference across the cavity is (say) 6° C. Then, from Figure 6, for a 12 mm airspace $h_c \simeq 1.6$. Substituting this into the expression for q we get

$$q = 18(0.1 + 0.006 + 0.625 + 0.006 + 0.05)^{-1}$$

$$= \frac{18}{0.787} = 22.87.$$

Thus from our equation for $\theta_2 - \theta_3$ we have

$$\theta_2 - \theta_3 = \frac{22.87}{1.6} \simeq 14.3.$$

Looking again at Figure 6, the value of h_c for a temperature drop of 14.3° C is about 1.7. This gives

$$q = 18(0.1 + 0.006 + 0.588 + 0.006 + 0.05)^{-1}$$

$$= \frac{18}{0.750} = 24.00,$$

and

$$\theta_2 - \theta_3 = \frac{24}{1.7} = 14.1.$$

So, within the accuracy attainable with Figure 6, the answer is that there is a heat transfer rate of about 24 W through each square metre of window, and a temperature drop of 14° C across the air gap. (We are only looking for a reasonable estimate.)

Thus in one hour, energy transmitted through 1 m^2 of glass $\simeq 24 \times 3600$ J.

$$\text{Cost} \simeq 16 \times 24 \times 3600 \times \frac{10^{-8}}{1.055} = 0.013 \text{ pence.}$$

So double glazing has reduced the cost by a factor of $4\frac{1}{2}$ or so. Notice again that the calculation of the heat transfer rate across the cavity took into account both conduction and convection, by way of the combined convection–conduction coefficient obtained from Figure 6. The same applies to the exercise below, for which the combined convection–conduction coefficient is part of the data.

Exercise 6

A brick wall of a house consists of two layers of brick separated by an air space. Work out the U value for this cavity wall and compare your answer with that to Exercise 1 on p. 18. Assume the following data.

Measurement data: each layer of brick is 0.12 m thick.

Heat transfer data:

$$\begin{aligned} \kappa &= 0.7 \text{ for brick,} && \text{units W m}^{-1}\,{}^\circ\text{C}^{-1} \\ h &= 8 \text{ for the inside face,} && \\ &= 19 \text{ for the outside face,} && \text{units W m}^{-2}\,{}^\circ\text{C}^{-1} \\ h_c &= 1.6 \text{ for the cavity.} && \end{aligned}$$

[*Solution on p. 28*]

Summary of Section 4

1. Convective heat transfer in a fluid can be modelled using the following assumptions:

(i) the temperature in the bulk fluid is uniform, say θ_f;

(ii) if the fluid meets a plane solid surface, with area A and temperature θ_s, then the rate of heat transfer from surface to fluid is $hA(\theta_s - \theta_f)$, where h is the **convective heat transfer coefficient**.

2. The equation for heat transfer through a wall or window with area A is

$$q = AU(\theta_{in} - \theta_{out})$$

where θ_{in} and θ_{out} are the bulk air temperatures inside and outside the building and U is a constant known as the ***U* value**. U values can be calculated by adding up the formulae for the differences between adjacent temperatures as in the solution to Example 1.

3. Convective heat transfer from a surface such as a pipe wall can be increased by means of fins. The heat transfer in the fin can be modelled by assuming that the temperature in the fin depends only on its distance x from the pipe wall, and

then considering convection from each element of the fin and conduction from one element to another. This leads to the differential equation

$$\frac{d^2\theta}{dx^2} = \frac{2h}{\kappa b}(\theta - \theta_a)$$

where h is the convective heat transfer coefficient, κ is the thermal conductivity of the material of the fin, b is its thickness and θ_a is the temperature of the air.

4. The heat transfer properties of a layer of fluid (such as the air space in double glazing) are complicated, and best modelled by a **combined heat transfer coefficient** h_c, which depends both on the thickness of the layer and on the temperature difference across it (see Figure 6).

5 End of unit problems

Read all the following five problems and attempt as many as you have time for *without* first looking at the solutions. Then compare your attempts with the solutions. Then read again the problems you have not attempted and read carefully through their solutions. In this way you may learn something from *all* the problems. (Solutions on pages 29–30.)

Problem 1

2 kg of water at 80° C are added to 5 kg of water at 20° C and allowed to mix thoroughly. Neglecting any energy exchanged with the container or with the surroundings, estimate the final temperature of the water.

Problem 2

A storage vessel may be modelled as a hollow sphere of uniform material. The internal and external radii are r_1 and r_2 respectively. The temperatures of the inner and outer spherical surfaces are θ_1 and θ_2 respectively and the thermal conductivity of the material of the vessel is κ.

(i) Show that the steady-state heat transfer rate through the wall of the vessel is given by

$$q = 4\pi\kappa(\theta_1 - \theta_2)\left\{\frac{1}{r_1} - \frac{1}{r_2}\right\}^{-1}.$$

(ii) Show that if the storage vessel is made of steel, and $\theta_2 = 20°$ C, $\theta_1 = 15°$ C and $r_1 = 0.1$ m, then without insulation the heat transfer rate through the wall of the vessel cannot be kept below about 340 W however large the outside radius r_2 is made.

(Hint: the surface area of a sphere of radius r is $4\pi r^2$.)

Problem 3

In the solution to Example 2 of Subsection 4.1 we found that the rate of heat transfer from the hot water inside an unlagged pipe to the cooler air outside is

$$q = 2\pi l(\theta_{in} - \theta_{out})\left\{\frac{1}{r_1 h_{in}} + \frac{1}{\kappa}\log_e(r_2/r_1) + \frac{1}{r_2 h_{out}}\right\}^{-1}.$$

Derive the corresponding expression for the heat transfer rate q_{lag} from a pipe fitted with a layer of lagging which has a convective heat transfer coefficient h_{lag} and coefficient of thermal conductivity κ_1, the new outside radius being r_3. Hence derive the condition which the value of h_{lag} must satisfy if the new heat transfer rate is to be less than the original one.

Problem 4

This problem concerns the content of Subsection 4.2. By integrating the right-hand side of Equation (4) on p. 21 and using the result of Exercise 4 on p. 22, estimate the total heat transfer rate per metre of axial length from the fin specified in Exercise 5 to the atmosphere. Compare your result with the solution to Exercise 5(ii) and comment on the difference.

Problem 5

The average rate at which energy is received from the sun at the surface of the earth is about 1347 W m^{-2}. Estimate the surface temperature of the sun, assuming (i) that the constant μ in the radiation equation is proportional to the surface area A of the radiating body, (ii) that for the sun $\mu = A \times 56.7 \times 10^{-9}$ (the units being W K^{-4}), (iii) that the ratio of the radius of the earth's orbit to the sun's radius is 216, and (iv) that the earth's radius may be neglected.

(Hint: the beams of radiation from the sun's surface spread out as indicated in Figure 1.)

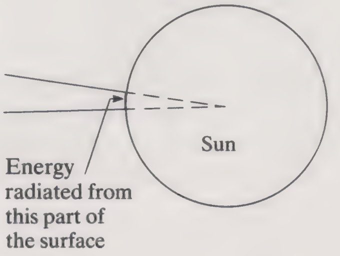


Figure 1

Appendix 1: Solutions to the exercises

Solutions to the exercises in Section 1

1. From Table 1, $c = 4200$, and from Equation (1),

$$\begin{aligned} E_2 - E_1 &= mc(\theta_2 - \theta_1) \\ &= 2 \times 4200 \times (80 - 20) \\ &= 504\,000 \\ &= 5.04 \times 10^5, \end{aligned}$$

i.e. there is an *increase* in internal energy of 5.04×10^5 J.

2. The energy output of the element is 2000 J s^{-1}. Thus the time to produce 5.04×10^5 J will be

$$\begin{aligned} &\frac{5.04 \times 10^5}{2 \times 10^3} \text{ seconds} \\ &= 252 \text{ seconds} \\ &= 4 \text{ minutes } 12 \text{ seconds.} \end{aligned}$$

3. Some of the energy is used to heat the kettle, and some is used to heat the air surrounding the kettle. The rate at which energy is used to heat the water is therefore less than was assumed in Exercise 2, and so the time obtained is an *under*estimate.

Solutions to the exercises in Section 2

1. From Table 2, $\kappa = 0.7$. Substituting this and the values $A = 1$, $b = 0.24$ into Equation (2), we obtain the rate of heat transfer through 1 m^2 of wall:

$$\begin{aligned} q &= 0.7 \times 1 \times \left(\frac{18 - 10}{0.24}\right) \\ &= 23.3. \end{aligned}$$

Thus the rate of heat transfer through 1 m^2 of wall is 23.3 W.

2. The temperature gradient is $\dfrac{-8}{0.24} = -33.3$, the units being °C m^{-1}. Let the temperature at 100 mm from the inner face be θ °C. Then

$$\frac{\theta - 18}{0.1} = -33.3;$$

$$\begin{aligned} \theta &= 18 - (0.1 \times 33.3) \\ &\simeq 14.7, \end{aligned}$$

and the required temperature is 14.7° C.

3. From Table 2, $\kappa = 1.0$ for glass. Substituting this and the values for l, θ_1, θ_2, r_1 and r_2 given in the exercise into Equation (5), we obtain the rate of heat transfer through one metre length of pipe wall:

$$\begin{aligned} q &= \frac{2\pi \times 1.0 \times 1 \times (90 - 50)}{\log_e (40/25)} \\ &= 535. \end{aligned}$$

Thus the rate of heat transfer through 1 m length of pipe wall is 535 W.

4. Let the temperature be θ °C at radius $x = 0.032$. Then by Equation (4),

$$\theta = -\frac{535}{2\pi \times 1.0 \times 1} \log_e (0.032) + C,$$

and we can find C because we know that $\theta = 90$ at $x = r_1 = 0.025$,.

Thus,

$$\begin{aligned} 90 &= -\frac{535}{2\pi \times 1.0 \times 1} \log_e (0.025) + C \\ &= 314 + C, \end{aligned}$$

so $C = -224$, giving

$$\theta = 69.1,$$

i.e. the temperature in the pipe wall at a radius of 32 mm is 69.1° C.

An alternative method would be to observe that Equation (5) applies to that part of the pipe wall lying between $r_1 = 25$ mm and $x = 32$ mm, so that

$$q = 535 = \frac{2\pi \times 1.0 \times 1 \times (90 - \theta)}{\log_e (32/25)}.$$

Solutions to the exercises in Section 3

1. In this case, the length of the bar (0.2 m) is the quantity corresponding to the thickness b in Equation (2) on p. 11 of Subsection 2.3. The cross-section area in m^2 is

$$\begin{aligned} A &= \frac{\pi}{4} \times 0.012^2 \\ &= 1.131 \times 10^{-4}. \end{aligned}$$

The rate of heat flow in watts is $q = 1.12$.

Since $\theta_1 - \theta_2 = 10$, Equation (2) gives

$$1.12 = \frac{\kappa \times 1.131 \times 10^{-4} \times 10}{0.2},$$

giving

$$\kappa = 198,$$

i.e. the thermal conductivity of the material is 198 W m^{-1} °C^{-1}.

2. (i) $-250°\text{C} = (273 - 250)\,\text{K} = 23\,\text{K}$.

(ii) $3°\text{C} = (273 + 3)\,\text{K} = 276\,\text{K}$.

3. For radiation, $q = \mu\theta_A^4 = \mu(\theta + 273)^4$.

Assuming that the filament dissipates its energy by radiation only,

$$40 = \mu(\theta + 273)^4.$$

If $\theta = 2000$, then

$$\mu = \frac{40}{(2273)^4} = \frac{40}{2.67 \times 10^{13}} = 1.5 \times 10^{-12},$$

the appropriate units being W K^{-4}.

Solutions to the exercises in Section 4

1. $$U = \left\{\frac{1}{h_{\text{in}}} + \frac{b}{\kappa} + \frac{1}{h_{\text{out}}}\right\}^{-1}$$
$$= \left\{\frac{1}{8} + \frac{0.24}{0.7} + \frac{1}{19}\right\}^{-1}$$
$$= 1.92,$$

the units being the same as those of h, namely $\text{W m}^{-2}\,°\text{C}^{-1}$.

2. For the rate of heat transfer through $1\,\text{m}^2$ of wall we put $A = 1$ in Equation (2), and obtain

$$q = 1 \times 1.92 \times (21 - 0) = 40.3.$$

Thus the rate of heat transfer through $1\,\text{m}^2$ of wall is 40.3 W.

The thermal resistance is $\dfrac{1}{U} = \dfrac{1}{1.92} = 0.521$, the units here being $\text{m}^2\,°\text{C W}^{-1}$.

3. Putting $\dfrac{1}{h_{\text{in}}} = \dfrac{1}{h_{\text{out}}} = 0$, we obtain

$$U = \left(\frac{0.24}{0.7}\right)^{-1}$$
$$= 2.92.$$

The percentage error in the heat transfer rate will be the same as the percentage error in U, namely

$$\frac{2.92 - 1.92}{1.92} \times 100 = 52.1\,\%.$$

4. By Equation (9) on p. 22,

$$\theta - \theta_a = Fe^{\lambda x} + Ge^{-\lambda x}.$$

At $x = 0$, $\theta = \theta_0$ (one of the boundary conditions), so

$$\theta_0 - \theta_a = F + G = T. \qquad (1)$$

Also,

$$\frac{d}{dx}(\theta - \theta_a) = \frac{d\theta}{dx} = \lambda Fe^{\lambda x} - \lambda Ge^{-\lambda x}.$$

At $x = w$, $\dfrac{d\theta}{dx} = 0$ (this is the other boundary condition).

Hence,

$$\lambda Fe^{\lambda w} - \lambda Ge^{-\lambda w} = 0. \qquad (2)$$

From (2),

$$Fe^{\lambda w} = Ge^{-\lambda w},$$

i.e.

$$Fe^{2\lambda w} = G. \qquad (3)$$

Substituting this expression for G into (1),

$$F(1 + e^{2\lambda w}) = T.$$

Thus, $F = \dfrac{T}{1 + e^{2\lambda w}}$,

and using (3) again,

$$G = \frac{Te^{2\lambda w}}{1 + e^{2\lambda w}}.$$

Hence, $\theta - \theta_a = \dfrac{T}{1 + e^{2\lambda w}}\{e^{\lambda x} + e^{(2\lambda w - \lambda x)}\}$

as required.

5.

(i) $$\lambda = \sqrt{\frac{2h}{\kappa b}}$$
$$= \sqrt{\frac{2 \times 10}{380 \times 0.002}} = 5.130.$$

We also have $T = 150 - 20 = 130$, $w = 0.05$, and midway along the fin x has the value 0.025. Hence,

$$\theta - \theta_a = \frac{130}{1 + \exp(2 \times 5.13 \times 0.05)} \times$$
$$\times\{\exp(5.13 \times 0.025) + \exp(2 \times 5.13 \times 0.05 - 5.13 \times 0.025)\}$$
$$= \frac{130}{1 + 1.670}\{1.137 + 1.469\}$$
$$= 127.$$

Since $\theta_a = 20$, we conclude that the temperature half-way along the fin is 147° C.

(ii) Assuming 147° C to be the mean temperature of the fin, the heat transfer rate is given by

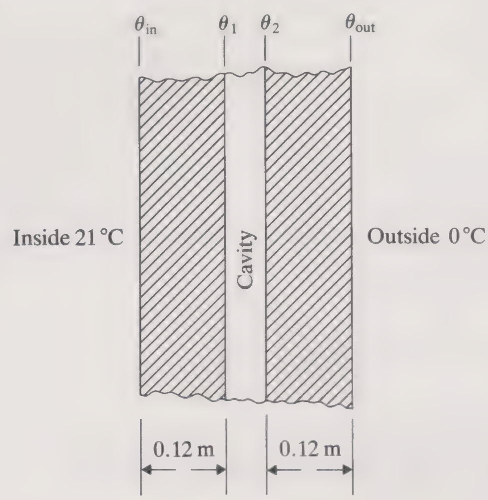
$$q = 2wlh(\theta_{\text{mean}} - \theta_a)$$

(where the factor 2 occurs because the fin has two sides). To find the heat transfer per metre of axial length we put $l = 1$. Thus

$$q = 2 \times 0.05 \times 1 \times 10 \times (147 - 20)$$
$$= 127,$$

so that the heat transfer rate from the fin is 127 W per metre length.

6. A section of the cavity wall is shown below.



Using the notation of this figure, the energy transfer per second per m^2 is given by five equations:

for the inside layer of brick,

$$q = 8(21 - \theta_{\text{in}}), \qquad \text{i.e. } 21 - \theta_{\text{in}} = \frac{q}{8},$$

$$q = 0.7\left(\frac{\theta_{\text{in}} - \theta_1}{0.12}\right), \qquad \text{i.e. } \theta_{\text{in}} - \theta_1 = \frac{0.12q}{0.7};$$

for the cavity,

$$q = 1.6(\theta_1 - \theta_2), \qquad \text{i.e. } \theta_1 - \theta_2 = \frac{q}{1.6};$$

for the outside layer of brick,

$$q = 0.7\left(\frac{\theta_2 - \theta_{\text{out}}}{0.12}\right), \qquad \text{i.e. } \theta_2 - \theta_{\text{out}} = \frac{0.12q}{0.7};$$

$$q = 19(\theta_{\text{out}} - 0), \qquad \text{i.e. } \theta_{\text{out}} = \frac{q}{19}.$$

Adding:

$$21 = q\left\{\frac{1}{8} + \frac{0.12}{0.7} + \frac{1}{1.6} + \frac{0.12}{0.7} + \frac{1}{19}\right\}.$$

$$\text{Thus,} \quad U = \left\{\frac{1}{8} + \frac{0.12}{0.7} + \frac{1}{1.6} + \frac{0.12}{0.7} + \frac{1}{19}\right\}^{-1}$$

$$= 0.873.$$

That is, the U value is 0.873 $\text{W m}^{-2}\,{}^\circ\text{C}^{-1}$.

In Exercise 1, for the same total thickness of brick but without the cavity we have a U value of 1.92 $\text{W m}^{-2}\,{}^\circ\text{C}^{-1}$, so the cavity has reduced the heat transfer rate to less than half of the former value. Filling the cavity with plastic foam can reduce the U value to about 0.5 $\text{W m}^{-2}\,{}^\circ\text{C}^{-1}$.

Appendix 2: Solutions to the problems

1. Heat transfer will cease when all the water is at the same temperature, say θ °C.

Then the change in the internal energy of the warmer water is $\{2 \times 4200 \times (\theta - 80)\}$ J, and the change in the internal energy of the cooler water is $\{5 \times 4200 \times (\theta - 20)\}$ J. Since no energy is given to or received from the surroundings, the total internal energy of the water must remain constant, and hence the total change in internal energy must be zero. Thus,

$$\{2 \times 4200 \times (\theta - 80)\} + \{5 \times 4200 \times (\theta - 20)\} = 0.$$

Cancelling out the factor of 4200 and collecting terms, we get

$$7\theta = 260,$$

$$\theta = \tfrac{260}{7} = 37.1.$$

Thus the final temperature will be 37.1° C.

2.

(i) The temperature at any point in the wall is determined solely by its distance from the centre of the sphere. Hence the temperature gradient in the steady state is $\dfrac{d\theta}{dx}$ where θ is the temperature at a point whose distance from the centre of the sphere is x. The rate of heat transfer through the entire spherical area at distance x from the centre is given by

$$q = -\kappa \times 4\pi x^2 \times \frac{d\theta}{dx}.$$

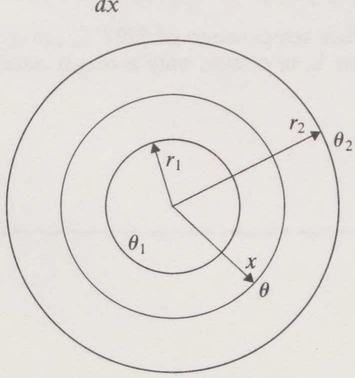


Figure 1

Hence

$$\frac{d\theta}{dx} = \frac{-q}{4\pi\kappa x^2}.$$

In the steady state, q is a constant, so this equation can be solved by direct integration to give the general solution

$$\theta = \int \frac{-q}{4\pi\kappa x^2}\,dx$$

$$= \frac{q}{4\pi\kappa x} + C$$

where C is a constant of integration.

Thus since $\theta = \theta_1$ when $x = r_1$ and $\theta = \theta_2$ when $x = r_2$, we have

$$\theta_1 = \frac{q}{4\pi\kappa r_1} + C,$$

$$\theta_2 = \frac{q}{4\pi\kappa r_2} + C,$$

and subtracting,

$$\theta_1 - \theta_2 = \frac{q}{4\pi\kappa}\left(\frac{1}{r_1} - \frac{1}{r_2}\right),$$

so that

$$q = 4\pi\kappa(\theta_1 - \theta_2)\left\{\frac{1}{r_1} - \frac{1}{r_2}\right\}^{-1}.$$

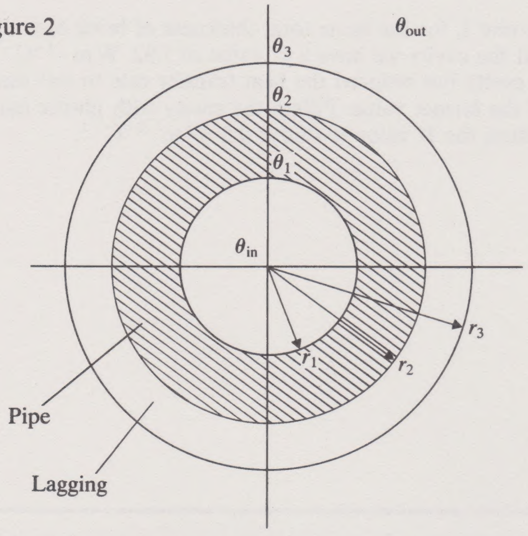
(ii) As r_2 increases, the heat transfer rate derived above decreases, to a limiting value of

$$4\pi\kappa(\theta_1 - \theta_2)\left(\frac{1}{r_1}\right)^{-1} = 4\pi\kappa r_1(\theta_1 - \theta_2).$$

For the given figures (and taking κ for steel = 54 from Table 2 on p. 10), this gives a heat transfer rate of 339 W. Thus the heat transfer rate cannot be kept below about 340 W.

3. A section of the lagged pipe is sketched below.

Figure 2



We have four equations for heat transfer from a length l of this pipe:

$$q_{lag} = 2\pi r_1 l h_{in}(\theta_{in} - \theta_1), \quad \text{i.e. } \theta_{in} - \theta_1 = \frac{q_{lag}}{2\pi r_1 l h_{in}};$$

$$q_{lag} = \frac{2\pi\kappa l(\theta_1 - \theta_2)}{\log_e(r_2/r_1)}, \quad \text{i.e. } \theta_1 - \theta_2 = \frac{q_{lag}\log_e(r_2/r_1)}{2\pi\kappa l};$$

$$q_{lag} = \frac{2\pi\kappa_1 l(\theta_2 - \theta_3)}{\log_e(r_3/r_2)}, \quad \text{i.e. } \theta_2 - \theta_3 = \frac{q_{lag}\log_e(r_3/r_2)}{2\pi\kappa_1 l};$$

$$q_{lag} = 2\pi r_3 l h_{lag}(\theta_3 - \theta_{out}), \quad \text{i.e. } \theta_3 - \theta_{out} = \frac{q_{lag}}{2\pi r_3 l h_{lag}}.$$

Adding,

$$\theta_{in} - \theta_{out} = \frac{q_{lag}}{2\pi l}\left\{\frac{1}{r_1 h_{in}} + \frac{\log_e(r_2/r_1)}{\kappa} + \frac{\log_e(r_3/r_2)}{\kappa_1} + \frac{1}{r_3 h_{lag}}\right\},$$

and so

$$q_{lag} = 2\pi l(\theta_{in} - \theta_{out}) \times \\ \times\left\{\frac{1}{r_1 h_{in}} + \frac{\log_e(r_2/r_1)}{\kappa} + \frac{\log_e(r_3/r_2)}{\kappa_1} + \frac{1}{r_3 h_{lag}}\right\}^{-1}.$$

Thus, to meet the condition $q_{lag} < q$ we must have

$$\frac{1}{r_1 h_{in}} + \frac{\log_e(r_2/r_1)}{\kappa} + \frac{\log_e(r_3/r_2)}{\kappa_1} + \frac{1}{r_3 h_{lag}} \\ > \frac{1}{r_1 h_{in}} + \frac{\log_e(r_2/r_1)}{\kappa} + \frac{1}{r_2 h_{out}};$$

that is,

$$\frac{\log_e(r_3/r_2)}{\kappa_1} + \frac{1}{r_3 h_{lag}} > \frac{1}{r_2 h_{out}};$$

$$\frac{1}{r_3 h_{lag}} > \frac{1}{r_2 h_{out}} - \frac{\log_e(r_3/r_2)}{\kappa_1};$$

$$h_{lag} < \frac{1}{r_3}\left\{\frac{1}{r_2 h_{out}} - \frac{\log_e(r_3/r_2)}{\kappa_1}\right\}^{-1}.$$

This means that if h_{lag} is too high the heat transfer rate of the lagged pipe can, at least in principle, be greater than that of the unlagged pipe!

4. By Equation (4) on p. 21, the rate of heat transfer from an element of the fin of width δx is given by

$$h \times 2l\delta x \times (\theta - \theta_a).$$

Thus the total heat transfer rate for the whole fin is

$$\int_0^w 2lh(\theta - \theta_a)\,dx$$
$$= 2lh\int_0^w (\theta - \theta_a)\,dx$$
$$= 2lh\int_0^w \frac{T}{1 + e^{2\lambda w}}\{e^{\lambda x} + e^{(2\lambda w - \lambda x)}\}\,dx \quad \text{(by Exercise 4)}$$
$$= \frac{2lhT}{1 + e^{2\lambda w}}\left[\frac{e^{\lambda x}}{\lambda} - \frac{e^{(2\lambda w - \lambda x)}}{\lambda}\right]_0^w$$
$$= \frac{2lhT}{\lambda(1 + e^{2\lambda w})}\{(e^{\lambda w} - e^{\lambda w}) - (1 - e^{2\lambda w})\}$$
$$= \frac{2lhT(e^{2\lambda w} - 1)}{\lambda(e^{2\lambda w} + 1)}.$$

For the figures in Exercise 5 this gives, for $l = 1$:

$$\frac{2 \times 10 \times 130 \times (1.670 - 1)}{5.130 \times (1.670 + 1)} \simeq 127$$

giving a heat transfer rate of 127 W per metre length. This agrees (to three figures) with the result of Exercise 5.

5. Below is a sketch (not to scale) of a beam of energy radiated from the sun to the earth.

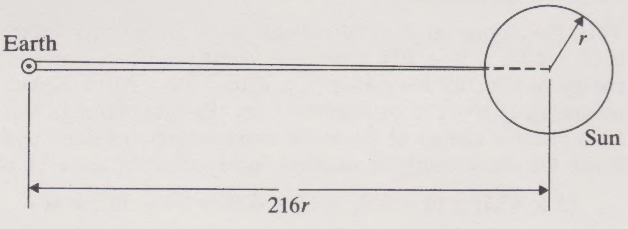


Figure 3

If the beam has a cross-sectional area of $1\,\text{m}^2$ at the earth's surface, it will have a cross-sectional area of $(\frac{1}{216})^2\,\text{m}^2$ at the surface of the sun. Thus the rate of radiation from $(\frac{1}{216})^2\,\text{m}^2$ of the sun's surface is 1347 W. Hence,

$$1347 = \mu \times (\theta + 273)^4$$

where $\mu = (\frac{1}{216})^2 \times 56.7 \times 10^{-9}$. This gives

$$(\theta + 273)^4 = \frac{1347 \times (216)^2 \times 10^8}{5.67}$$
$$= 11\,083\,886 \times 10^8.$$

Thus,

$$\theta + 273 = 57.7 \times 10^2 = 5770,$$

giving a surface temperature of 5497° C (or approximately 5500° C). This is, of course, only a rough estimate.

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